followed into the proximal end of the spinal cord. In ornithorhynchus, Ziehen could not satisfy himself about the existence of any pyramidal system. Prof. Kölliker believes he can distinguish a pyramidal decussation in ornithorhynchus and that the fibres of it plunge mainly into the dorsal column of the spinal cord (as in the rat and guinea-pig) and not into the lateral column, as in the generality of mammals. But the description he gives is a very unsatisfactory one, and no mention is made of his data for discrimination between the undoubtedly existent fillet decussation and the equivocally existent pyramidal. Moreover, he admits that he finds in ornithorhynchus no trace of longitudinal fibres passing anywhere along the pons. In arguing for the existence of a pyramidal system, he omits mention of what to most minds forms the strongest argument yet adducible, namely, that, as shown by Prof. C. J. Martin, of Melbourne, excitation of a certain region of the cerebral cortex of ornithorhynchus evokes movement of the crossed fore-limb.

The large extent and specially sentient character of oral-facial surface in ornithorhynchus prepares the observer for a large recipient nucleus in the bulb to subserve the huge sensory root of the trigeminus. This root and its recipient nucleus form a huge projection either side the bulb—the tuberculum quinti, well shown in a figure reproduced from Elliot Smith. Prof. Kölliker finds fibres of this root traceable to the nuclei of the *hypoglossus, vago-glossopharyngeus* and *abducens,* as well as to that of the trigeminus itself. From the recipient nucleus of trigeminus he traces fibres to the median fillet, and so to the optic thalamus.   C. S. S.

DIVERGENT SERIES.

Leçons sur les Séries Divergentes. Par Émile Borel. Pp. viii+184. (Paris: Gauthier-Villars, 1901.) Price fr. 4.50.

TO make the object of this work intelligible, it is necessary to recall a few facts concerning infinite series in general. Suppose we have a sequence

$$u_1, u_2, u_3, \ldots u_n, \ldots \qquad \text{(U)}$$

where $u_1, u_2,$ &c., are analytical expressions constructed by a definite rule. Let $s_n = u_1 + u_2 + \ldots + u_n$; then we have a derived analytical sequence

$$s_1, s_2, s_3, \ldots s_n, \ldots; \qquad \text{(S)}$$

this is a definite analytical entity, and its properties are implicitly fixed by those of the former sequence. The expressions u_n, s_n are, of course, functions of n; we may suppose, for simplicity, that, besides this, they involve, in addition to definite numerical constants, a single analytical variable, x. If we assign to x a numerical value, S becomes an arithmetical sequence, and three principal cases arise, according to the behaviour of s_n when n increases indefinitely. If s_n converges to a definite limit s we say that this is the sum of the series

$$u_1 + u_2 + u_3 + \ldots, \text{ and write } s = \sum_1^\infty u_n;$$

but the ultimate value of s may be either indeterminate or infinite. In the second case $\sum u_n$ has no definite meaning; in the third we may say, if we like, that $\sum u_n$ is infinite, but this

infinite sum is not a quantity with which we can operate, and presents no special interest.

When the series $\sum u_n$ and $\sum v_n$ are absolutely convergent we can add and multiply them according to the rules

$$\sum u_n + \sum v_n = \sum (u_n + v_n)$$
$$\sum u_n \times \sum v_n = u_1 v_1 + (u_1 v_2 + u_2 v_1) + (u_1 v_3 + u_2 v_2 + u_3 v_1) + \ldots$$
$$= \sum_n \sum_r (u_r v_{n+1-r}):$$

now the sequences

$$u_1 + v_1, \ u_2 + v_2, \ \ldots \ u_n + v_n, \ \ldots \qquad \text{(A)}$$
$$u_1 v_1, \ u_1 v_2 + u_2 v_1, \ \ldots \ u_1 v_n + u_2 v_{n-1} + \ldots + u_n v_1, \ldots \qquad \text{(B)}$$

can be constructed, whether or not the sequences $(u_1, u_2, \ldots)$ and $(v_1, v_2, \ldots)$ are convergent; the question therefore arises whether it is possible, even when the series $\sum u_n, \sum v_n$ are divergent, to associate with the sequences $(u_1, u_2 \ldots)$ and $(v_1, v_2, \ldots)$ certain finite and determinate functions U, V in such a way that $U + V$ and UV may be *by the same rule of correspondence* associated with the sequences (A) and (B) above.

Among the various ways in which this can be done, M. Borel's method of exponential summation is particularly interesting. Briefly it is this: let

$$u(a) = u_0 + u_1 a + \frac{u_2 a^2}{2!} + \frac{u_3 a^3}{3!} + \ldots + \frac{u_n a^n}{n!} + \ldots$$

then the function

$$s = \int_0^\infty e^{-a} u(a) da$$

is defined to be the *exponential sum* of the series $u_0 + u_1 + u_2 + \ldots$. When $\sum u_n$ is convergent, s coincides with the sum in the ordinary sense; the important point is that s may be finite even when $\sum u_n$ is divergent; the series is then " exponentially summable "—*absolutely* so, if

$$\bar{s} = \int_0^\infty e^{-a} \mid u(a) \mid da$$

is a convergent integral. M. Borel proves that (in the case of absolute summability) if U, V are the exponential sums of $\sum u_n$ and $\sum v_n$, then $U + V$ and UV are the exponential sums of the series whose terms are given under (A) and (B) above; in other words, the formal laws of rational operation are satisfied. In a similar sense, an absolutely summable series may be differentiated any number of times.

As an example of the practical value of these results, suppose we have a differential equation $F(y, y', y'', \ldots) = 0$ in which $y, y', y'' \ldots$ enter rationally: then, if this is found to be formally satisfied by a series $\sum u_n$, which, although divergent in the ordinary sense, is exponentially summable, the exponential sum is actually a solution of the differential equation.

In Chapter iv. M. Borel applies the idea of exponential summation to an interesting problem in function-theory. Suppose we have a power-series

$$u_0 + u_1 x + u_2 x^2 + \ldots$$

which is convergent within a circle of finite radius, but divergent outside of it. Within the circle, this series defines a function of x, say $f(x)$; within the same region the series is exponentially summable, and its sum is $f(x)$. But the exponential sum may exist and be finite in a region *larger* than the circle of convergence of the power-series; in this case the exponential sum is an analytical continuation of $f(x)$ outside the circle, and the new region

of summability is shown to comprise an area bounded by a. (finite or infinite) number of straight lines, each of which goes through a critical point. This new region M. Borel calls the *polygon of summability*. An obvious question arises here; does the continuation of $f(x)$ obtained by exponential summation necessarily coincide with one obtained by other methods, for example Weierstrass's? In some cases it certainly does; for instance, when $f(x)$ is a rational function of x, or one branch of an algebraic function.

So far it has been supposed that the object of inquiry is in the first instance a series given by the law of construction of its terms; and the main result has been to show how, in certain cases where the series is divergent in the ordinary sense, it may be associated with a finite function, called its sum (in an extended sense), which the series so far represents that relations satisfied formally by the series are actually and arithmetically satisfied by its sum. But there is another side of the question which is of equal importance, especially from the practical point of view. We may have a function explicitly or implicitly defined by certain properties, and try to obtain a series which for purposes of computation or otherwise may be regarded as its equivalent. A typical illustration is afforded by the ordinary process of solving differential equations by series; here we have a uniform method which, if it does not fail altogether, leads to a power-series, formally satisfying the equation, but not necessarily convergent. Exponential summation, when it is applicable, enables us to obtain a solution from the merely formal equivalent. In this connection we have Poincaré's theory of asymptotic series, which is expounded by M. Borel in Chapter i. Independently of its convergence, the expansion

$$c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots + \frac{c_n}{x^n} + \ldots$$

is said to represent $f(x)$ asymptotically if

$$x^n \left[f(x) - c_0 - \frac{c_1}{x} - \frac{c_2}{x^2} - \ldots - \frac{c_n}{x^n} \right]$$

vanishes when x is infinite. Asymptotic expansions may be combined by the ordinary formal rules of rational operations, and the result is asymptotically equivalent to the corresponding combination of the functions represented. These considerations justify the use of semi-convergent series in computation; the classical example occurs in the theory of the gamma function. It must be carefully observed, however, that although the asymptotic expansion (if it exist) of a definite function is itself definite, we *cannot* infer the existence of a definite function corresponding to a given expansion $\Sigma c_n x^{-n}$: the reason of this rather paradoxical result is that innumerable functions (for instance e^{-x}) lead to an asymptotic expansion with zero coefficients throughout.

In Chapter ii. M. Borel discusses the results contained in Stieltjes's memoir (*Annales de la Faculté des Sciences de Toulouse*, tt. viii. ix.), and in Chapter v. deals with the polynomial expansions due to Mittag-Leffler. Interesting as they are, it seems hopeless to try to analyse these chapters within the compass of a review; they are, indeed, themselves of the nature of summaries, and will be best appreciated by those readers who accept M. Borel's invitation to consult the original memoirs. Attention may, however, be called to the author's

estimation of these researches. It is, in effect, that the memoir of Stieltjes, though of great originality and suggestiveness, is of restricted application and not likely to lead to a general theory; and that, on the other hand, while Mittag-Leffler's theory does not immediately afford a calculus of divergent series, in the proper sense of the term, it may very probably lead to one. It should be added that M. Borel himself has made substantial contributions to this theory of polynomial expansions; some of them appear for the first time in the present volume.

The fact is that most of the field traversed in this very attractive course is of recent discovery, and we cannot expect to be presented with a complete and symmetrical doctrine all at once. Let us be thankful that M. Borel, himself one of the pioneers on this novel route, has so clearly and impartially indicated the progress that has hitherto been made.   G. B. M.

OUR BOOK SHELF.

The Chemical Essays of Charles-William Scheele. Pp. xxx + 294. (London: Scott, Greenwood and Co., 1901.) Price 5*s*. net.

THIS is a reprint of Dr. Beddoes' translation of Scheele's essays, which was published in 1786 by John Murray and may still be picked up occasionally in second-hand book shops. The reproduction is faithful even to the mis-spelling of Priestley's name in Beddoes' preface. Between this preface and the essays, however, there now appears a memoir of the life and work of Scheele, written for the reissue by Mr. John Geddes McIntosh. Mr. McIntosh presumably has inspired the reissue of the essays, and if this will be the means of getting them more generally read by students of chemistry, he may so far prove a benefactor.

Of the essays themselves it is hardly necessary to say anything. The facts they establish belong for the most part to what is now very elementary chemistry and the phlogistic hypothesis with which the explanations are involved did not long outlive Scheele; but the spirit which breathes in these essays and the method they inculcate can never grow commonplace or antiquated.

The strict fidelity to experiment, the rare sagacity, the scrupulous and minute observation and the extraordinary experimental skill combine to make Scheele a model for all time. When we add to this the pathos of his early struggles, the simplicity of his blameless life and the nobility of his untimely death, there can be no wonder that Scheele is reckoned a hero among chemists.

It cannot be said that the memoir which accompanies these essays is worthy of the subject. Mr. McIntosh has apparently considerable enthusiasm for the solid virtues of Scheele and for the material outcome of Scheele's discoveries, but he shows little critical insight or literary taste. Speaking, for example, of the discovery of chlorine, he says: "Let us now glance at the radical errors of the French school, the chief of whom was Berthollet, the man who was the first to make practical application of Scheele's discovery, and, as is usually the case with such men, they propound a theory of their own, so that some at least of the merit, if not all of the original discovery, may descend upon their own mantle."

The violence here done to Berthollet, to the rules of English composition, and to a time-honoured metaphor is very remarkable.

On the following page it is stated to be "a well-known fact at the present day" that the product of distilling fluorspar and sulphuric acid in a glass retort is gaseous hydrofluosilicic acid.